

## ON SPECTRAL ASSIGNMENT FOR NEUTRAL TYPE SYSTEMS

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ABSTRACT. For a large class of linear neutral type systems the problem of eigenvalues and eigenvectors assignment is investigated, i.e. finding the system which has the given spectrum and almost all, in some sense, eigenvectors.

## 1. INTRODUCTION

One of central problems in control theory is the spectral assignment problem. This question is well investigated for linear finite dimensional systems. It is important to emphasize that the assignment of eigenvalues is not sufficient in several cases. One needs also the assignment of eigenvectors or of the geometric eigenstructure. For infinite dimensional problems (delay systems, partial derivative equations) the problem is much more complicated.

Our purpose is to investigate this kind of problems for a large class of neutral type systems given by the equation

$$(1.1) \quad \dot{z}(t) = A_{-1}\dot{z}(t-1) + \int_{-1}^0 A_2(\theta)\dot{z}(t+\theta)d\theta + \int_{-1}^0 A_3(\theta)z(t+\theta)d\theta,$$

where  $z(t) \in \mathbb{R}^n$  and  $A_{-1}, A_2, A_3$  are  $n \times n$  matrices. The elements of  $A_2$  and  $A_3$  taking values in  $L_2(-1, 0)$ . The neutral type term  $A_{-1}\dot{z}(t-1)$  consists on a simple delay, while the other include as multiple as distributed delays. The behavior of such systems can be described mainly by the algebraic and geometric properties of the spectrum of the matrix  $A_{-1}$  (cf. [4, 5]).

It is well known that then spectral properties of this system are described by the characteristic matrix  $\Delta(\lambda)$  given by

$$\Delta(\lambda) = \lambda I - \lambda e^{-\lambda} A_{-1} - \int_{-1}^0 \lambda e^{\lambda\theta} A_2(\theta)d\theta - \int_{-1}^0 e^{\lambda\theta} A_3(\theta)d\theta.$$

The eigenvalues are roots of the equation  $\det \Delta(\lambda) = 0$ . The eigenvectors of the system (more precisely of the functional operator model of the system) are expressed through the matrix  $\Delta(\lambda_k)$ , where  $\lambda_k$  is an eigenvalue. In fact the problem of an assignment of an infinite number of eigenvalues and eigenvectors is reduced to a problem of assignment of singular values and degenerating vectors of an entire matrix value function  $\Delta(\lambda)$ . It is remarkable [5] that the roots of  $\det \Delta(\lambda) = 0$  are quadratically close to a fixed set of complex number which are the logarithm of eigenvalues of the matrix  $A_{-1}$ . Moreover, the degenerating vectors of  $\Delta(\lambda_k)$  are also quadratically close to the eigenvectors of the matrix  $A_{-1}$ .

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*Date:* April 17, 2013.

*2010 Mathematics Subject Classification.* Primary 93C23, Secondary 93B60, 93B55.

*Key words and phrases.* Neutral type systems, eigenvalue assignment, eigenvector assignment.

This work was supported in part by the Polish Nat. Sci. Center, grant N N514 238 438.

In this paper we investigate an inverse problem:

*What conditions must satisfy a sequence of complex numbers  $\{\lambda\}$  and a sequence of vectors  $\{v\}$  in order to be a sequence of roots of the characteristic equation  $\det \Delta(\lambda) = 0$  and a sequence of degenerating vectors of the characteristic matrix  $\Delta(\lambda)$  of equation (1.1) respectively for some choice of matrices  $A_{-1}, A_2(\theta), A_3(\theta)$  ?*

One of the possible application of this problem is to investigate a vector moment problem via the solution of the exact controllability property for a corresponding neutral type system by a relation devlopped in [3].

The present paper is a detailed version of the short note published in Comptes Rendus Mathematiques [7].

## 2. OPERATOR FORM OF PERTURBATION

We consider neutral type systems of the form

$$(2.1) \quad \dot{z}(t) = A_{-1}\dot{z}(t-1) + \int_{-1}^0 A_2(\theta)\dot{z}(t+\theta)d\theta + \int_{-1}^0 A_3(\theta)z(t+\theta)d\theta,$$

where  $A_{-1}$  is a constant  $n \times n$  matrix,  $A_2, A_3$  are  $n \times n$  matrices whose elements belong to  $L_2[-1, 0]$ .

As it is shown in [4], [5] this system can be rewritten in the operator form

$$\frac{d}{dt} \begin{pmatrix} y \\ z_t(\cdot) \end{pmatrix} = \mathcal{A} \begin{pmatrix} y \\ z_t(\cdot) \end{pmatrix},$$

where  $\mathcal{A} : D(\mathcal{A}) \rightarrow M_2 = \mathbb{C}^n \times L_2([-1, 0], \mathbb{C}^n)$ ,

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} y \\ \varphi(\cdot) \end{pmatrix} \mid \varphi(\cdot) \in H^1([-1, 0], \mathbb{C}^n), y = \varphi(0) - A_{-1}\varphi(-1) \right\} \subset M_2,$$

and the operator  $\mathcal{A}$  is given by formula

$$\mathcal{A} \begin{pmatrix} y \\ \varphi(\cdot) \end{pmatrix} = \begin{pmatrix} \int_{-1}^0 A_2(\theta)\dot{\varphi}(\theta)d\theta + \int_{-1}^0 A_3(\theta)\varphi(\theta)d\theta \\ \frac{d\varphi}{d\theta}(\cdot) \end{pmatrix}.$$

This operator is noted  $\tilde{\mathcal{A}}$  instead of  $\mathcal{A}$  if  $A_2(\theta) = A_3(\theta) \equiv 0$ . The operator  $\tilde{\mathcal{A}}$  is defined on the same domain  $D(\mathcal{A})$ . One can consider that the state operator  $\mathcal{A}$  is a perturbation of the operator  $\tilde{\mathcal{A}}$ , namely

$$\mathcal{A} \begin{pmatrix} y \\ \varphi(\cdot) \end{pmatrix} = \tilde{\mathcal{A}} \begin{pmatrix} y \\ \varphi(\cdot) \end{pmatrix} + \begin{pmatrix} \int_{-1}^0 A_2(\theta)\dot{\varphi}(\theta)d\theta + \int_{-1}^0 A_3(\theta)\varphi(\theta)d\theta \\ 0 \end{pmatrix}.$$

Let  $\mathcal{B}_0 : \mathbb{C}^n \rightarrow M_2$  be given by

$$\mathcal{B}_0 y = \begin{pmatrix} y \\ 0 \end{pmatrix},$$

and  $\mathcal{P}_0 : D(\mathcal{A}) \rightarrow \mathbb{C}^n$  by

$$(2.2) \quad \mathcal{P}_0 \begin{pmatrix} y \\ \varphi(\cdot) \end{pmatrix} = \int_{-1}^0 A_2(\theta)\dot{\varphi}(\theta)d\theta + \int_{-1}^0 A_3(\theta)\varphi(\theta)d\theta.$$

Then  $\mathcal{A} = \tilde{\mathcal{A}} + \mathcal{B}_0\mathcal{P}_0$ . Denote by  $X_{\mathcal{A}}$  the set  $D(\mathcal{A})$  endowed with the graph norm. Let us show that  $\mathcal{P}_0$  browses the set of all linear bounded operators  $\mathcal{L}(X_{\mathcal{A}}, \mathbb{C}^n)$  as

$A_2(\cdot), A_3(\cdot)$  run over the set of  $n \times n$  matrices with components from  $L_2[-1, 0]$ . Indeed, an arbitrary linear operator  $Q$  from  $\mathcal{L}(X_{\mathcal{A}}, \mathbb{C}^n)$  can be presented as

$$\begin{aligned} Q \begin{pmatrix} y \\ \varphi(\cdot) \end{pmatrix} &= Q_1 y + Q_2 \varphi(\cdot) \\ &= Q_1(\varphi(0) - A_{-1}\varphi(-1)) + \int_{-1}^0 \widehat{A}_2(\theta) \dot{\varphi}(\theta) d\theta + \int_{-1}^0 \widehat{A}_3(\theta) \varphi(\theta) d\theta, \end{aligned}$$

where  $\widehat{A}_2(\cdot), \widehat{A}_3(\cdot)$  are  $(n \times n)$ -matrices with component from  $L_2[-1, 0]$  and  $Q_1$  is a  $(n \times n)$  matrix. Let us observe that

$$\begin{aligned} \varphi(-1) &= \int_{-1}^0 \theta \dot{\varphi}(\theta) d\theta + \int_{-1}^0 \varphi(\theta) d\theta, \\ \varphi(0) &= \int_{-1}^0 (\theta + 1) \dot{\varphi}(\theta) d\theta + \int_{-1}^0 \varphi(\theta) d\theta \end{aligned}$$

and denote

$$\begin{aligned} A_2(\theta) &= \widehat{A}_2(\theta) + (\theta + 1)Q_1 - \theta Q_1 A_{-1}, \\ A_3(\theta) &= \widehat{A}_3(\theta) + Q_1 - Q_1 A_{-1}. \end{aligned}$$

Then, with these notations, the operator  $Q$  may be written as

$$Q \begin{pmatrix} y \\ \varphi(\cdot) \end{pmatrix} = \int_{-1}^0 A_2(\theta) \dot{\varphi}(\theta) d\theta + \int_{-1}^0 A_3(\theta) \varphi(\theta) d\theta.$$

Hence formula (2.2) describes all the operators from  $\mathcal{L}(X_{\mathcal{A}}, \mathbb{C}^n)$ .

### 3. AN EQUATION FOR EIGENVALUES AND EIGENVECTORS OF THE CHARACTERISTIC MATRIX (SPECTRAL EQUATION)

Consider the operator  $\mathcal{A} = \widetilde{\mathcal{A}} + \mathcal{B}_0 \mathcal{P}_0$  and assume that  $\lambda_0$  is an eigenvalue of  $\mathcal{A}$  and  $x_0$  is a corresponding eigenvector, i.e.

$$(3.1) \quad (\widetilde{\mathcal{A}} + \mathcal{B}_0 \mathcal{P}_0)x_0 = \lambda_0 x_0.$$

Let us assume further that  $\lambda_0$  does not belong to spectrum of  $\widetilde{\mathcal{A}}$  and denote by  $R(\widetilde{\mathcal{A}}, \lambda_0) = (\widetilde{\mathcal{A}} - \lambda_0 I)^{-1}$ , with this notation (3.1) reads as

$$(3.2) \quad x_0 + R(\widetilde{\mathcal{A}}, \lambda_0) \mathcal{B}_0 \mathcal{P}_0 x_0 = 0.$$

Let us notice that  $v_0 = \mathcal{P}_0 x_0 \neq 0$ , because  $\lambda_0 \notin \sigma(\widetilde{\mathcal{A}})$ . Then applying operator  $\mathcal{P}_0$  to the left hand side of (3.2) we get

$$v_0 + \mathcal{P}_0 R(\widetilde{\mathcal{A}}, \lambda_0) \mathcal{B}_0 v_0 = 0$$

This equality means that  $\lambda_0$  is a point of singularity of the matrix-valued function  $F(\lambda) = I + \mathcal{P}_0 R(\widetilde{\mathcal{A}}, \lambda) \mathcal{B}_0$  and  $v_0$  is a vector degenerating  $F(\lambda_0)$  from the right.

Let  $w_0^*$  be a non-zero row such that

$$(3.3) \quad w_0^* F(\lambda_0) = 0.$$

In order to describe the vector  $w_0$ , let us find first another form for the matrix  $F(\lambda)$ . For any  $v \in \mathbb{C}^n$  we denote

$$R(\widetilde{\mathcal{A}}, \lambda) \mathcal{B}_0 v = \begin{pmatrix} y \\ \varphi(\cdot) \end{pmatrix}, \quad \lambda \notin \sigma(\widetilde{\mathcal{A}}),$$

Then

$$(\tilde{\mathcal{A}} - \lambda I) \begin{pmatrix} y \\ \varphi(\cdot) \end{pmatrix} = \mathcal{B}_0 v = \begin{pmatrix} v \\ 0 \end{pmatrix}.$$

This gives

$$v = -\lambda y, \quad \frac{d\varphi(\theta)}{d\theta} - \lambda \varphi(\theta) = 0,$$

and, as  $\begin{pmatrix} y \\ \varphi(\cdot) \end{pmatrix} \in D(\tilde{\mathcal{A}})$ , we obtain  $y = \varphi(0) - A_{-1}\varphi(-1)$ . Therefore

$$\varphi(\theta) = D e^{\lambda\theta}, \quad D \in \mathbb{C}^n,$$

and

$$v = -\lambda(I - e^{-\lambda} A_{-1})D.$$

Since the matrix  $(I - e^{-\lambda} A_{-1})$  is invertible ( $\lambda \notin \sigma(\tilde{\mathcal{A}})$ ), then we get

$$\varphi(\theta) = -\frac{e^{\lambda\theta}}{\lambda}(I - e^{-\lambda} A_{-1})^{-1}v,$$

and hence

$$R(\tilde{\mathcal{A}}, \lambda) \mathcal{B}_0 v = \begin{pmatrix} y \\ -\frac{e^{\lambda\theta}}{\lambda}(I - e^{-\lambda} A_{-1})^{-1}v \end{pmatrix}.$$

This formula and (2.2) implies

$$\begin{aligned} v + \mathcal{P}_0 R(\tilde{\mathcal{A}}, \lambda) \mathcal{B}_0 v &= \\ &= v - \left( \int_{-1}^0 e^{\lambda\theta} A_2(\theta) d\theta - \int_{-1}^0 \frac{e^{\lambda\theta}}{\lambda} A_3(\theta) d\theta \right) (I - e^{-\lambda} A_{-1})^{-1}v \end{aligned}$$

and hence

$$F(\lambda)v = \Delta(\lambda) \frac{(I - e^{-\lambda} A_{-1})^{-1}}{\lambda} v,$$

where

$$\Delta(\lambda) = \lambda I - \lambda e^{-\lambda} A_{-1} - \int_{-1}^0 \lambda e^{\lambda\theta} A_2(\theta) d\theta - \int_{-1}^0 e^{\lambda\theta} A_3(\theta) d\theta$$

is the characteristic matrix of equation (1.1). Thus, the equality (3.3) for  $\lambda_0 \notin \sigma(\tilde{\mathcal{A}})$  is equivalent to

$$w_0^* \Delta(\lambda_0) = 0.$$

Summarizing we obtain the following

**Proposition 3.1.** *Let  $\lambda_0$  do not belong to  $\sigma(\tilde{\mathcal{A}})$ . Then the pair  $(\lambda_0, w_0)$ ,  $w_0 \in \mathbb{C}^n$ ,  $w_0 \neq 0$ , satisfies equation (3.3) if and only if  $\lambda_0$  is a root of the characteristic equation*

$$\det \Delta(\lambda) = 0$$

*and  $w_0^*$  is a row-vector degenerating  $\Delta(\lambda_0)$  from the left, i.e.  $w_0^* \Delta(\lambda_0) = 0$ .*

Thus, one can consider the equation  $w^* F(\lambda) = 0$  as an equation whose roots  $(\lambda, w) = (\lambda_0, w_0)$  describe all eigenvalues and (right) eigenvectors of the characteristic matrix  $\Delta(\lambda)$ .

## 4. A COMPONENT-WISE REPRESENTATION OF SPECTRAL EQUATION

We recall spectral properties of operators  $\tilde{\mathcal{A}}$  and  $\tilde{\mathcal{A}}^*$  obtained in [4, 5]. We will assume that the matrix  $A_{-1}$  has a simple non-zero eigenvalues  $\mu_1, \mu_2, \mu_3, \dots, \mu_n$ . In this case the spectrum  $\sigma(\tilde{\mathcal{A}})$  consists of simple eigenvalues which we denote by

$$\tilde{\lambda}_k^m = \ln |\mu_m| + i(\text{Arg } \mu_m + 2\pi k), \quad m = 1, \dots, n, k \in \mathbb{Z},$$

and of eigenvalue  $\tilde{\lambda}_0 = 0$ . First assume  $\mu_m \neq 1$ ,  $m = 1, \dots, n$ . Then the corresponding to  $\tilde{\lambda}_k^m$  eigenvectors are of the form

$$(4.1) \quad \tilde{\varphi}_k^m = \begin{pmatrix} 0 \\ e^{\tilde{\lambda}_k^m \theta} y_m \end{pmatrix}, \quad k \in \mathbb{Z}, m = 1, \dots, n,$$

where  $y_1, \dots, y_n$  are eigenvectors of  $A_{-1}$  corresponding to  $\mu_1, \mu_2, \dots, \mu_n$ . The eigenspace corresponding to  $\tilde{\lambda}_0 = 0$  is  $n$  dimensional and its basis is

$$(4.2) \quad \tilde{\varphi}_j^0 = \begin{pmatrix} (1 - \mu_j) y_j \\ y_j \end{pmatrix}, \quad j = 1, \dots, n.$$

If some  $\mu_m$  say  $\mu_1$  equals 1, then  $\tilde{\lambda}_0^1 = \tilde{\lambda}_0 = 0$ . In that case the eigenspace corresponding to 0 is  $(n + 1)$ -dimensional and its basis consists of  $n$  eigenvectors  $\tilde{\varphi}_j^0$ ,  $j = 1, \dots, n$ , given by (4.2), and one rootvector

$$\tilde{\varphi}_0^1 = \begin{pmatrix} y_1 \\ \theta y_1 \end{pmatrix}, \quad \tilde{\mathcal{A}} \tilde{\varphi}_0^1 = \tilde{\varphi}_1^0.$$

All the vectors  $\tilde{\varphi}_k^m$  given by (4.1), except of  $\tilde{\varphi}_1^0$ , are still eigenvectors of  $\tilde{\mathcal{A}}$  and correspond to eigenvalues  $\tilde{\lambda}_k^m$ . In both cases the family  $\Phi = \{\tilde{\varphi}_k^m\} \cup \{\tilde{\varphi}_j^0\}$  forms a Riesz basis in the space  $M_2$ . Denote by  $\Psi = \{\tilde{\psi}_k^m\} \cup \{\tilde{\psi}_j^0\}$  the bi-orthogonal basis to  $\Phi$ . Then

$$(4.3) \quad \tilde{\psi}_k^m = \begin{pmatrix} z_m / \overline{\tilde{\lambda}_k^m} \\ e^{-\overline{\tilde{\lambda}_k^m} \theta} z_m \end{pmatrix}, \quad m = 1, \dots, n, k \in \mathbb{Z}, \tilde{\lambda}_0^1 \neq 0,$$

where  $\overline{\tilde{\lambda}}$  is the complex conjugate of  $\tilde{\lambda}$  and

$$\tilde{\psi}_0^1 = \begin{pmatrix} z_1 \\ 0 \end{pmatrix}$$

if  $\tilde{\lambda}_0^1 = 0$ , here  $z_m$  are eigenvectors of matrix  $A_{-1}^*$  such that

$$\langle y_i, z_j \rangle = \delta_{ij}.$$

It is easy to see that  $\tilde{\psi}_k^m$  are eigenvectors of the operator  $\mathcal{A}^*$  corresponding to eigenvalues  $\overline{\tilde{\lambda}_k^m}$ .

Our nearest goal is to rewrite the matrix  $F(\lambda_0)$  in the basis  $\Phi$ . Let us first rewrite the expression  $R(\tilde{\mathcal{A}}, \lambda_0) \mathcal{B}_0$ . We present  $\mathcal{B}_0$  as a matrix  $\mathcal{B}_0 = (b_1^0, b_2^0, \dots, b_n^0)$  with infinite columns  $b_i^0$  which are vectors from  $M_2$  of the form

$$b_i^0 = \begin{pmatrix} e_i \\ 0 \end{pmatrix}, \quad i = 1, \dots, n,$$

where  $e_i$  is the canonical basis of  $\mathbb{C}^n$ . Then

$$b_i^0 = \sum_{k,m} \langle b_i^0, \tilde{\psi}_k^m \rangle \tilde{\varphi}_k^m + \sum_j \langle b_i^0, \tilde{\psi}_j^0 \rangle \tilde{\varphi}_j^0, \quad i = 1, \dots, n,$$

and in the case  $\mu_m \neq 1$ ,  $m = 1, \dots, n$ , we have

$$R(\tilde{\mathcal{A}}, \lambda_0) b_i^0 = \sum_{k,m} \frac{1}{\tilde{\lambda}_k^m - \lambda_0} \langle b_i^0, \tilde{\psi}_k^m \rangle \tilde{\varphi}_k^m - \sum_j \frac{1}{\lambda_0} \langle b_i^0, \tilde{\psi}_j^0 \rangle \tilde{\varphi}_j^0, \quad i = 1, \dots, n.$$

Taking into account the form of eigenvectors (4.3) we find that for all  $m = 1, \dots, n$

$$\left( \langle b_1^0, \tilde{\psi}_k^m \rangle, \dots, \langle b_n^0, \tilde{\psi}_k^m \rangle \right) = \frac{1}{\tilde{\lambda}_k^m} (\langle e_1, z_m \rangle, \dots, \langle e_n, z_m \rangle) = z_m^* / \tilde{\lambda}_k^m.$$

So we obtain

(4.4)

$$R(\tilde{\mathcal{A}}, \lambda_0) \mathcal{B}_0 = \sum_{k,m} \frac{1}{\tilde{\lambda}_k^m - \lambda_0} \times \frac{1}{\tilde{\lambda}_k^m} \tilde{\varphi}_k^m z_m^* - \sum_j \frac{1}{\lambda_0} \tilde{\varphi}_j^0 \left( \langle b_1^0, \tilde{\psi}_j^0 \rangle, \dots, \langle b_n^0, \tilde{\psi}_j^0 \rangle \right),$$

if all the numbers  $\tilde{\lambda}_k^m \neq 0$ . If  $\mu_1 = 1$  and  $\tilde{\lambda}_0^1 = 0$  then in the sum (4.4) the term corresponding to  $k = 1$ ,  $m = 1$ , is replaced by

$$\left( \frac{1}{\tilde{\lambda}_0^1 - \lambda_0} \tilde{\varphi}_0^1 - \frac{1}{(\tilde{\lambda}_0^1 - \lambda_0)^2} \tilde{\varphi}_1^0 \right) z_1^*.$$

In what follows, we shall use the notation

$$\tilde{\beta}_k^m = \begin{cases} \tilde{\lambda}_k^m, & \tilde{\lambda}_k^m \neq 0, \\ 1, & \tilde{\lambda}_k^m = 0, \end{cases} \quad m = 1, \dots, n, \quad k \in \mathbb{Z}.$$

Let us observe that the expression  $w_0^* \mathcal{P}_0$  is a linear bounded functional on the space  $X_{\mathcal{A}}$ , i.e.  $w_0^* \mathcal{P}_0 \in \mathcal{L}(X_{\mathcal{A}}, \mathbb{C})$ . The representation of  $w_0^*$  in the basis  $z_j$  is as follows:

$$w_0^* = \sum_j \alpha_j z_j^*.$$

Consider now  $n$  functionals  $z_j^* \mathcal{P}_0 \in \mathcal{L}(X_{\mathcal{A}}, \mathbb{C})$ ,  $j = 1, \dots, n$ . One can decompose them in the basis  $\Psi$ :

$$z_j^* \mathcal{P}_0 = \sum_{k,m} p_{k,m}^j \tilde{\psi}_k^m + \sum_i \tilde{p}_i^j \tilde{\psi}_i^0,$$

where

$$(4.5) \quad \sum_k \left| \frac{p_{k,m}^j}{\tilde{\beta}_k^m} \right|^2 < \infty, \quad m = 1, \dots, n.$$

In the sequel, we shall assume  $\tilde{p}_i^j = 0$ ,  $i = 1, \dots, n$ . This means that from now we consider perturbations  $\mathcal{P}_0$  satisfying the condition

$$(4.6) \quad \int_{-1}^0 A_3(\theta) d\theta = 0.$$

Then we have

$$w_0^* \mathcal{P}_0 = \sum_j \alpha_j z_j^* \mathcal{P}_0 = \sum_j \alpha_j \sum_{k,m} p_{k,m}^j \tilde{\psi}_k^m.$$

From this relation and expression (4.4) we obtain the equality

$$\begin{aligned} w_0^* \mathcal{P}_0 \mathcal{R}(\tilde{\mathcal{A}}, \lambda_0) \mathcal{B}_0 &= \sum_m \left\langle \sum_k \frac{1}{\tilde{\beta}_k^m} \times \frac{1}{\tilde{\lambda}_k^m - \lambda_0} \tilde{\varphi}_k^m, \sum_j \alpha_j \sum_{k_0, m_0} \tilde{\beta}_{k_0, m_0}^j \tilde{\psi}_{k_0}^{m_0} \right\rangle z_m^* \\ &= \sum_j \alpha_j \sum_{k_0, m_0} \frac{p_{k_0, m_0}^j}{\tilde{\beta}_{k_0}^{m_0}} \times \frac{1}{\tilde{\lambda}_{k_0}^{m_0} - \lambda_0} z_m^*. \end{aligned}$$

With these notations, the equation (3.3) reads

$$0 = w_0^* \left( I + \mathcal{P}_0 \mathcal{R}(\tilde{\mathcal{A}}, \lambda_0) \mathcal{B}_0 \right) = \sum_m \alpha_m z_m^* + \sum_m \left( \sum_{j, k} \alpha_j \frac{p_{k, m}^j}{\tilde{\beta}_k^m} \times \frac{1}{\tilde{\lambda}_k^m - \lambda_0} \right) z_m^*.$$

Thus, the condition for a pair  $(\lambda_0, w_0)$  to satisfy the spectral equation can be rewritten in the form of the following system of  $n$  equations:

$$(4.7) \quad \alpha_m = - \sum_{\substack{k \in \mathbb{Z} \\ j=1, \dots, n}} \alpha_j \left( \frac{p_{k, m}^j}{\tilde{\beta}_k^m} \times \frac{1}{\tilde{\lambda}_k^m - \lambda_0} \right), \quad m = 1, \dots, n,$$

where for any fixed couple  $m, j$ , the **needed**  $n$ -tuple  $\{p_{k, m}^j\}$  satisfies (4.5).

## 5. CONDITIONS FOR SPECTRAL ASSIGNMENT

Now we discuss the following question:

*What conditions must satisfy a sequence of complex numbers  $\{\lambda\}$  and a sequence of vectors  $\{v\}$  in order to be a sequence of roots of the characteristic equation  $\det \Delta(\lambda) = 0$  and a sequence of degenerating vectors of the characteristic matrix  $\Delta(\lambda)$  of equation (1.1) respectively for some choice of matrices  $A_{-1}, A_2(\theta), A_3(\theta)$ ?*

We will assume that the corresponding operator  $\mathcal{A}$  has simple eigenvalues only. Let us remember that we assumed earlier that all eigenvalues of matrix  $A_{-1}$  are also simple. Then one can enumerate those eigenvalues as  $\{\lambda_k^m\} \cup \{\lambda_j^0\}$ ,  $m, j = 1, \dots, n$ ;  $k \in \mathbb{Z}$ , where (see [5, Theorem 1]) the sequence  $\{\lambda_k^m\}$  satisfies

$$(5.1) \quad \sum_{k, m} \left| \lambda_k^m - \tilde{\lambda}_k^m \right|^2 < \infty.$$

Denote by  $\{\varphi_k^m\} \cup \{\varphi_j^0\}$ ,  $m, j = 1, \dots, n$ ;  $k \in \mathbb{Z}$ , corresponding eigenvectors of  $\mathcal{A}$ . Then (see [5, Lemma 13, Theorem 15]) these vectors form a Riesz basis in  $M_2$  which is quadratically close to the basis  $\{\tilde{\varphi}_k^m\} \cup \{\tilde{\varphi}_j^0\}$  if we assume that the corresponding elements have the same norm. Eigenvectors  $\{\varphi\}$  has the form

$$\varphi = \begin{pmatrix} v - e^{-\lambda} A_{-1} v \\ e^{\lambda \theta} v \end{pmatrix}$$

with  $\Delta(\lambda)v = 0$ . Therefore the fact that the basis  $\{\varphi\}$  and  $\{\tilde{\varphi}\}$  are quadratically close implies the condition

$$(5.2) \quad \sum_k \|v_k^m - y_m\|^2 < \infty, \quad m = 1, \dots, n,$$

where  $\Delta(\lambda_k^m)v_k^m = 0$ ,  $m = 1, \dots, n$ ;  $k \in \mathbb{Z}$ , and  $y_m$  are eigenvectors of  $A_{-1}$  corresponding to  $\mu_m$  as in (4.1). If we apply the same arguments for the dual system

$$(5.3) \quad \dot{z}(t) = A_{-1}^* \dot{z}(t-1) + \int_{-1}^0 A_2^*(\theta) \dot{z}(t+\theta) d\theta + \int_{-1}^0 A_3^*(\theta) z(t+\theta) d\theta,$$

then we obtain the symmetric condition

$$(5.4) \quad \sum_k \|w_k^m - z_m\|^2 < \infty, \quad m = 1, \dots, n,$$

where  $(w_k^m)^* \Delta(\lambda_k^m) = 0$ ,  $m = 1, \dots, n$ ;  $k \in \mathbb{Z}$ .

Our further goal is to show that conditions (5.1), (5.4) or (5.1), (5.2) are almost sufficient for couples of sequences  $\{\lambda\}, \{w\}$  or  $\{\lambda\}, \{v\}$  to be spectral ones for the system (1.1).

Let us consider a sequence of different complex numbers  $\{\lambda_{k_0}^{m_0}\}$ ,  $m_0 = 1, \dots, n$ ;  $k_0 \in \mathbb{Z}$ , satisfying (5.1). We also assume that the index numbering of  $\{\lambda\}$  is such that if  $\lambda_{k_0}^{m_0} = \tilde{\lambda}_k^m$  then  $m_0 = m$ ,  $k_0 = k$ . To begin with, however, we put  $\lambda_{k_0}^{m_0} \neq \tilde{\lambda}_k^m$ , for all  $k, k_0 \in \mathbb{Z}$ ,  $m, m_0 = 1, \dots, n$ .

Let now  $\{\lambda_{k_0}^{m_0}\}$  be simple eigenvalues of operator  $\mathcal{A} = \tilde{\mathcal{A}} + \mathcal{B}_0 \mathcal{P}_0$ , where  $\mathcal{P}_0$  is given by (2.2), in which matrix  $A_3(\theta)$  satisfies (4.6). Then  $\Delta(\lambda_{k_0}^{m_0}) = 0$  and let  $\{w_{k_0}^{m_0}\}$  be a sequence of the left degenerating vectors of  $\Delta(\lambda_{k_0}^{m_0})$ , i.e.

$$(w_{k_0}^{m_0})^* \Delta(\lambda_{k_0}^{m_0}) = 0.$$

We assume that the sequence  $w_{k_0}^{m_0}$  satisfies (5.4). For all indices  $m_0 = 1, \dots, n$ ;  $k_0 \in \mathbb{Z}$ , consider decompositions

$$(w_{k_0}^{m_0})^* = \sum_{j=1}^n \alpha_{jm_0}^{k_0} z_j^*.$$

Then condition (5.4) is equivalent to

$$(5.5) \quad \sum_{k_0} |\alpha_{mm_0}^{k_0}|^2 < \infty, \quad m \neq m_0, \quad \sum_{k_0} |\alpha_{mm}^{k_0} - 1|^2 < \infty, \quad m, m_0 = 1, \dots, n.$$

Let us rewrite now relations (4.7) for  $\lambda_0 = \lambda_{k_0}^{m_0}$  and  $w_0 = w_{k_0}^{m_0}$ .

We now consider the space  $\ell_2$  of infinite sequences (columns) indexed as  $\{a_k\}_{k \in \mathbb{Z}}$  with a scalar product defined by  $\langle \{a_k\}, \{b_k\} \rangle = \sum_k a_k \overline{b_k}$ . From the relation (4.5) we obtain that vectors

$$p_m^j = - \left\{ \frac{\overline{p}_{k,m}^j}{\overline{\lambda}_k^m - \lambda_{k_0}^{m_0}} \right\}_{k \in \mathbb{Z}, j, m=1, \dots, n}$$

belong to  $\ell_2$ . One can also easily see that

$$\left\{ \frac{1}{\overline{\lambda}_k^m - \lambda_{k_0}^{m_0}} \right\}_{k \in \mathbb{Z}} \in \ell_2, \quad m, m_0 = 1, \dots, n; \quad k_0 \in \mathbb{Z}.$$

Then, putting  $\lambda_0 = \lambda_{k_0}^{m_0}$  and  $w_0 = w_{k_0}^{m_0}$  in the equations (4.7), we obtain

$$(5.6) \quad \alpha_{m,m_0}^{k_0} = \sum_{j=1}^n \alpha_{jm_0}^{k_0} \left\langle \left\{ \frac{1}{\overline{\lambda}_k^m - \lambda_{k_0}^{m_0}} \right\}, p_m^j \right\rangle, \quad m, m_0 = 1, \dots, n; \quad k_0 \in \mathbb{Z}.$$

Now we would like to rewrite relation (5.6) in a vector-matrix abstract form. In order to do that, we introduce a more convenient notation. Denote

$$\alpha_{mm_0} = \{\alpha_{mm_0}^{k_0}\}_{k_0 \in \mathbb{Z}},$$

$m, m_0 = 1, \dots, n$ . By  $S_{mm_0} = \{s_{k_0 k}^{mm_0}\}_{k, k_0 \in \mathbb{Z}}$ ,  $m, m_0 = 1, \dots, n$ , we denote infinite matrices with elements

$$s_{k_0 k}^{mm_0} = \frac{1}{\tilde{\lambda}_k^m - \lambda_{k_0}^{m_0}},$$

and by  $A_{jm_0}$ ,  $j, m_0 = 1, \dots, n$ , infinite diagonal matrices

$$A_{jm_0} = \text{diag} \left\{ \alpha_{jm_0}^{k_0} \right\}_{k_0 \in \mathbb{Z}}.$$

With these notations relations (5.6) can be rewritten as

$$(5.7) \quad \sum_{j=1}^n A_{jm_0} S_{mm_0} p_m^j = \alpha_{mm_0},$$

$m, m_0 = 1, \dots, n$ . Now let us fix index  $m$  and consider  $n$  equations (5.7) with this index and  $m_0 = 1, 2, \dots, n$ . Consider another infinite diagonal matrix

$$\Lambda_m = \text{diag} \left\{ \tilde{\lambda}_k^m - \lambda_k^m \right\}_{k \in \mathbb{Z}},$$

and multiply both sides of the  $m$ -th equality (5.7) (for  $m_0 = m$ ) by this matrix from the left. This gives the following system of equalities

$$(5.8) \quad \begin{cases} \sum_{j=1}^n A_{jm_0} S_{mm_0} p_m^j = \alpha_{mm_0}, & m_0 = 1, \dots, n, \ m_0 \neq m, \\ \sum_{j=1}^n A_{jm} \Lambda_m S_{mm} p_m^j = \Lambda_m \alpha_{mm}, \end{cases}$$

where we used the fact that diagonal matrices commute :  $\Lambda_m A_{jm} = A_{jm} \Lambda_m$ . Finally, we introduce block matrix operators

$$D_m = \begin{bmatrix} A_{11} S_{m1} & \dots & A_{m1} S_{m1} & \dots & A_{n1} S_{m1} \\ \vdots & \ddots & \vdots & & \vdots \\ A_{1m} \Lambda_m S_{mm} & \dots & A_{mm} \Lambda_m S_{mm} & \dots & A_{nm} \Lambda_m S_{mm} \\ \vdots & & \vdots & \ddots & \vdots \\ A_{1n} S_{mn} & \dots & A_{mn} S_{mn} & \dots & A_{nn} S_{mn} \end{bmatrix}, \ m = 1, \dots, n,$$

and present (5.8) in the form

$$D_m \begin{bmatrix} p_m^1 \\ \vdots \\ p_m^m \\ \vdots \\ p_m^n \end{bmatrix} = \begin{bmatrix} \alpha_{m1} \\ \vdots \\ \Lambda_m \alpha_{mm} \\ \vdots \\ \alpha_{mn} \end{bmatrix}.$$

Let us observe that both vectors  $(p_m^1, \dots, p_m^n)^T$  and  $(\alpha_{m1} \dots \Lambda_m \alpha_{mm} \dots \alpha_{mn})^T$  belong to  $\ell_2^n = \underbrace{\ell_2 \times \ell_2 \times \dots \times \ell_2}_{n \text{ times}}$  (see (4.5),(5.5)). Therefore the system (5.8) is

solvable if and only if the vector  $(\alpha_{m1} \dots \Lambda_m \alpha_{mm} \dots \alpha_{mn})^T$  belongs to the image of operator  $D_m$  as an operator from  $\ell_2^n$  to  $\ell_2^n$ . In the sequel we show that, for all sequences  $\{\lambda\}$  satisfying, (5.1) and for almost all sequences  $\{w\}$  satisfying (5.4), operators  $D_m, m = 1, \dots, n$ , are bounded and with bounded inverse operators from  $\ell_2^n$  to  $\ell_2^n$ . This means that the spectral assignment problem is solvable. In the further argument we use the following

**Proposition 5.1.** *Let  $\{\lambda_k\}_{k \in \mathbb{Z}}$  be a sequence such that*

$$\sum_{k \in \mathbb{Z}} |\lambda_k - a + i(b + 2\pi k)|^2 < \infty,$$

*for some  $a, b \in \mathbb{R}$ . Then the family  $\{e^{\lambda_k t}\}_{k \in \mathbb{Z}}$  forms a Riesz basis in  $L_2(0, 1)$ .*

There are several ways to prove this classic result (see [1]). It may be obtained, for example, from the Paley-Wiener theorem [2] and Lemma II.4.11 [1].

Next we prove the following preliminary result.

**Lemma 5.2.** *1. For  $m \neq m_o$  operators  $S_{mm_o}$  are bounded as operators from  $\mathcal{L}(\ell_2)$  and have bounded inverses. 2.  $\Lambda_m S_{mm}$  is a bounded operator from  $\mathcal{L}(\ell_2)$  and has a bounded inverse.*

*Proof.* Let  $\{\varphi_k\}, \{\tilde{\varphi}_k\}, k \in \mathbb{Z}$ , be two Riesz basis of a Hilbert space  $H$  and let  $R$  be a bounded operator with a bounded inverse, such that  $R\varphi_k = \tilde{\varphi}_k, k \in \mathbb{Z}$ .

For  $f \in H$  we have

$$f = \sum_j a_j \varphi_j, \quad Rf = \sum_j a_j R\varphi_j$$

Then

$$R\varphi_j = \tilde{\varphi}_j = \sum_k c_{jk} \varphi_k = \sum_k \langle \tilde{\varphi}_j, \psi_k \rangle \varphi_k, \quad j \in \mathbb{Z},$$

where  $\{\psi_k\}_{k \in \mathbb{Z}}$  is the bi-orthogonal with respect basis to  $\{\varphi_k\}_{k \in \mathbb{Z}}$ . Hence

$$Rf = \sum_j a_j \sum_k \langle \tilde{\varphi}_j, \psi_k \rangle \varphi_k = \sum_k \sum_j a_j \langle \tilde{\varphi}_j, \psi_k \rangle \varphi_k = \sum_k b_k \varphi_k,$$

where  $b_k = \sum_j a_j \langle \tilde{\varphi}_j, \psi_k \rangle$ .

This means that the infinite matrix  $\hat{R}$  corresponding to  $R$  in the basis  $\{\varphi_k\}$  is of the form

$$\hat{R} = \{\hat{r}_{kj} = \langle \tilde{\varphi}_j, \psi_k \rangle\}_{\substack{k \in \mathbb{Z} \\ j \in \mathbb{Z}}},$$

where

$$\hat{R} \begin{bmatrix} \vdots \\ a_{-1} \\ a_0 \\ a_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ b_{-1} \\ b_0 \\ b_1 \\ \vdots \end{bmatrix}, \quad \{a_j\}, \{b_j\} \in \ell_2.$$

Let now  $H = L_2(0, 1)$  and  $\{\tilde{\varphi}_k\}_{k \in \mathbb{Z}}$  be a Riesz basis of the form  $\tilde{\varphi}_k = e^{\tilde{\lambda}_k^m t}$  for some  $m = 1, \dots, n$ . Let now  $\{\varphi_k\}$  be a Riesz basis which is bi-orthogonal to

$\{\psi_k = e^{-\overline{\lambda_k^{m_0} t}}\}_{k \in \mathbb{Z}}$  for some  $m_0 = 1, \dots, n$  (the fact that  $\{\psi_k\}_{k \in \mathbb{Z}}$  is a Riesz basis of  $L_2(0, 1)$  follows from Proposition 5.1). One has

$$\begin{aligned} \langle \tilde{\varphi}_j, \psi_k \rangle &= \int_{-1}^0 e^{\tilde{\lambda}_j^m t} e^{-\lambda_k^{m_0} t} dt = \frac{1}{\tilde{\lambda}_j^m - \lambda_k^{m_0}} (e^{\tilde{\lambda}_j^m - \lambda_k^{m_0}} - 1) \\ &= \frac{1}{\tilde{\lambda}_j^m - \lambda_k^{m_0}} (\mu_m e^{-\lambda_k^{m_0}} - 1), \end{aligned}$$

i.e.  $\hat{R} = \{\hat{r}_{kj}\}_{\substack{k \in \mathbb{Z} \\ j \in \mathbb{Z}}}$ , where

$$\hat{r}_{kj} = s_{kj} (\mu_m e^{-\lambda_k^{m_0}} - 1), \quad k, j \in \mathbb{Z}.$$

Thus

$$\hat{R} = \varepsilon_{mm_0} S_{mm_0},$$

where  $\varepsilon_{mm_0}$  is the infinite matrix

$$\varepsilon_{mm_0} = \begin{bmatrix} \ddots & \dots & \dots & \dots & \dots \\ \vdots & \mu_m e^{-\lambda_{-1}^{m_0}} - 1 & 0 & 0 & \vdots \\ \vdots & 0 & \mu_m e^{-\lambda_0^{m_0}} - 1 & 0 & \vdots \\ \vdots & 0 & 0 & \mu_m e^{-\lambda_1^{m_0}} - 1 & \vdots \\ \dots & \dots & \dots & \dots & \ddots \end{bmatrix}.$$

Hence, we have the following alternative:

1. If  $m_0 \neq m$ , then the sequence  $\{\mu_m e^{-\lambda_k^{m_0}} - 1\}_{k \in \mathbb{Z}}$  is bounded and separated from 0, i.e.  $\varepsilon_{mm_0} : \ell_2 \rightarrow \ell_2$  is a bounded operator with a bounded inverse. Hence,

$$(5.9) \quad S_{mm_0} = \varepsilon_{mm_0}^{-1} \hat{R}.$$

2. If  $m = m_0$ , then  $\mu_m e^{-\lambda_k^m} \rightarrow 1, k \rightarrow \infty$ , moreover

$$\begin{aligned} \mu_m e^{-\lambda_k^m} - 1 &= e^{\tilde{\lambda}_k^m - \lambda_k^m} - 1 \\ &= \left( 1 + (\tilde{\lambda}_k^m - \lambda_k^m) + \dots + \frac{(\tilde{\lambda}_k^m - \lambda_k^m)^s}{s!} + \dots \right) - 1 \\ &= (\tilde{\lambda}_k^m - \lambda_k^m) \left( 1 + \bar{o}(\tilde{\lambda}_k^m - \lambda_k^m) \right). \end{aligned}$$

Therefore,

$$\varepsilon_{mm} = \Lambda_m Q_m = Q_m \Lambda_m,$$

where  $Q_m = \text{diag} \left( 1 + \bar{o}(\tilde{\lambda}_k^m - \lambda_k^m) \right)_{k \in \mathbb{Z}}$  has a bounded inverse, so

$$(5.10) \quad \Lambda_m S_{mm} = Q_m^{-1} \hat{R}$$

From (5.9), (5.10) it follows that  $S_{mm_0}$ ,  $m \neq m_0$  and  $\Lambda_m S_{mm}$  are bounded and have bounded inverse.  $\square$

**Remark 5.3.** In our previous consideration we assumed implicitly that our sequences  $\{\lambda_{k_0}^{m_0}\}_{k_0 \in \mathbb{Z}}$  are different from  $\{\tilde{\lambda}_k^m\}_{k \in \mathbb{Z}}$ , i.e.  $\tilde{\lambda}_k^m \neq \lambda_{k_0}^{m_0}$  for all  $k, k_0 \in \mathbb{Z}$ ,  $m, m_0 \in \{1, \dots, n\}$  in particular  $\lambda_k^m \neq \tilde{\lambda}_k^m$  for every  $k \in \mathbb{Z}$ ,  $m \in \{1, \dots, n\}$ .

Now let us allow  $\lambda_k^m = \tilde{\lambda}_k^m$  for some indices  $k \in I \subset \mathbb{Z}$ . Note that in this case the operators  $S_{mm_0}$ ,  $m_0 \neq m$ , are still well-defined and the operator  $\Lambda_m S_{mm}$  can be well-defined as well if we define its components as limits of correspondent components when  $\lambda_k^m \rightarrow \tilde{\lambda}_k^m$ ,  $k \in I$ . This means that for  $k \in I$  all non-diagonal elements of the  $k$ -th line of  $\Lambda_m S_{mm}$  equal 0 and the diagonal elements equal 1. Besides,  $S_{mm_0}$ ,  $m_0 \neq m$  and  $\Lambda_m S_{mm}$  remains bounded and with a bounded inverse  $\ell_2 \rightarrow \ell_2$  operators since formulas (5.9) (5.10) remain true also when  $\lambda_k^m = \tilde{\lambda}_k^m$ . Finally if we consider the dependence  $\Lambda_m S_{mm}$  of the sequence  $\{\lambda_k^m\}_{k \in \mathbb{Z}}$ , one can easily prove that

$$\Lambda_m S_{mm}(\{\lambda_k^m\}) \rightarrow \Lambda_m S_{mm}(\{\tilde{\lambda}_k^m\})$$

as

$$\sum_k |\lambda_k^m - \tilde{\lambda}_k^m|^2 \rightarrow 0.$$

In other words this means that operators  $\Lambda_m S_{mm}$  and, as a consequence also its inverse operators  $(\Lambda_m S_{mm})^{-1}$ , depend continuously of sequence  $\{\lambda_k^m\}_{k \in \mathbb{Z}}$  on the set

$$\left\{ \{\lambda_k^m\} : \sum_k |\lambda_k^m - \tilde{\lambda}_k^m|^2 < \infty \right\}.$$

Now we are ready to prove our main results on the spectral assignment.

**Theorem 5.4.** *Let  $\mu_1, \mu_2, \dots, \mu_n$  be different nonzero complex numbers and  $z_1, z_2, \dots, z_n$  be nonzero  $n$ -dimensional linear independent vectors. Denote*

$$\tilde{\lambda}_k^m = \ln |\mu_m| + i(\text{Arg } \mu_m + 2\pi k), \quad m = 1, \dots, n, \quad k \in \mathbb{Z}.$$

*Let us consider an arbitrary sequence of different complex numbers  $\{\lambda_k^m\}_{k \in \mathbb{Z}, m=1, \dots, n}$  such that*

$$\sum_k |\lambda_k^m - \tilde{\lambda}_k^m|^2 < \infty, \quad m = 1, \dots, n.$$

*Then there is a small enough  $\varepsilon > 0$  such for any sequence of nonzero vectors  $\{d_k^m\}_{k \in \mathbb{Z}, m=1, \dots, n}$  satisfying*

$$\sum_k \|d_k^m - z_m\|^2 < \varepsilon, \quad m = 1, \dots, n$$

*one can choose matrices  $A_{-1}, A_2(\theta), A_3(\theta)$  such that for the system (1.1), with these matrices, the following two conditions hold:*

- i) *all the numbers  $\{\lambda_k^m\}$  are roots of the characteristic equation  $\det \Delta(\lambda_k^m) = 0$ ,  $k \in \mathbb{Z}, m = 1, \dots, n$ ,*
- ii)  *$d_k^m$  are right degenerating vectors for  $\Delta(\lambda_k^m)$  :  $d_k^{m*} \Delta(\lambda_k^m) = 0$ ,  $m = 1, \dots, n; k \in \mathbb{Z}$ .*

*Such a choice is unique if we put the following additional condition on matrix  $A_3(\theta)$ :*

$$(C) \quad \int_{-1}^0 A_3(\theta) d\theta = 0.$$

*Proof.* First we denote by  $A_{-1}$  the matrix uniquely defined by the relations:

$$z_m^* A_{-1} = \mu_m z_m^*, \quad m = 1, \dots, n,$$

and denote by  $\{y_j\}_{j=1, \dots, n}$  the bi-orthogonal basis with respect to  $\{z_j\}_{j=1, \dots, n}$  in  $\mathbb{C}^n$ . Then the corresponding operator  $\tilde{\mathcal{A}}$ , generated by the matrix  $A_{-1}$ , has eigenvalues  $\tilde{\lambda}_k^m$ ,  $m = 1, \dots, n$ ,  $k \in \mathbb{Z}$ ;  $\tilde{\lambda}_0 = 0$  and possesses the Riesz basis of eigenvectors of  $\{\tilde{\varphi}_k^m\}_{k \in \mathbb{Z}} \cup \{\tilde{\varphi}_j^0\}_{j=1, \dots, n}$ . In the case when all  $\mu_m \neq 1$ ,  $m = 1, \dots, n$ , the corresponding eigenvectors of  $\tilde{\mathcal{A}}$  are

$$(5.11) \quad \begin{cases} \tilde{\varphi}_k^m = \begin{pmatrix} 0 \\ e^{\tilde{\lambda}_k^m \theta} y_m \end{pmatrix}, & m = 1, \dots, n; \quad k \in \mathbb{Z}, \quad \tilde{\mathcal{A}} \tilde{\varphi}_k^m = \tilde{\lambda}_k^m \tilde{\varphi}_k^m, \\ \tilde{\varphi}_j^0 = \begin{pmatrix} y_j - A_{-1} y_j \\ y_j \end{pmatrix}, & j = 1, \dots, n; \quad \tilde{\mathcal{A}} \tilde{\varphi}_j^0 = 0. \end{cases}$$

In the case  $\mu_1 = 1$ , the vector

$$\varphi_0^1 = \begin{pmatrix} A_{-1} y_j \\ \theta y_j \end{pmatrix}, \quad j = 1, \dots, n;$$

is a generalized eigenvector of  $\tilde{\mathcal{A}}$  corresponding to  $\tilde{\lambda}_0$  and the other  $\tilde{\varphi}_k^m$ ,  $\tilde{\varphi}_j^0$  are given by formula (5.11). Let us show that there is a choice of a bounded operator  $\mathcal{P}_0 : X_{\tilde{\mathcal{A}}} \rightarrow \mathbb{C}^n$  (or equivalently a choice of matrices  $A_2(\theta)$ ,  $A_3(\theta)$ ) such that

$$(5.12) \quad \lambda_{k_0}^{m_0} \in \sigma(\tilde{\mathcal{A}} + \mathcal{B}_0 \mathcal{P}_0),$$

or equivalently  $\det \Delta(\lambda_{k_0}^{m_0}) = 0$ , and

$$(5.13) \quad d_{k_0}^{m_0*} \Delta(\lambda_{k_0}^{m_0}) = 0, \quad m = 1, \dots, n; \quad k \in \mathbb{Z}.$$

We represent vectors  $d_{k_0}^{m_0}$  in the basis  $\{z_j\}_{j=1, \dots, n}$ , namely

$$d_{k_0}^{m_0} = \sum_{m=1}^n \alpha_{mm_0}^{k_0} z_m, \quad k_0 \in \mathbb{Z}; \quad m, m_0 = 1, \dots, n.$$

With these notations the condition

$$\sum_{k_0} \|d_{k_0}^m - z_m\|^2 < \infty, \quad m = 1, \dots, n$$

implies

$$\sum_{k_0} |\alpha_{mm_0}^{k_0}|^2 < \infty, \quad m \neq m_0; \quad \sum_{k_0} |\alpha_{mm}^{k_0} - 1|^2 < \infty, \quad m = 1, \dots, n.$$

and these sums tend to zero as

$$(5.14) \quad \sum_{k_0} \|d_{k_0}^m - z_m\|^2 \rightarrow 0.$$

Therefore, under condition (5.14) operators  $D_m$ ,  $m = 1, \dots, n$  tend to block diagonal operators

$$\hat{D}_m = \begin{bmatrix} S_{m1} & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \Lambda_m S_{mm} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & S_{mn} \end{bmatrix}$$

which have a bounded inverse due to Lemma 5.2. This means that for a small enough  $\varepsilon > 0$ , the inequality  $\sum_k \|d_k^m - z_m\|^2 < \varepsilon$  implies that operators  $D_m$ ,  $m = 1, \dots, n$  have bounded inverse. If numbers  $\{\lambda_k^m\}$  are different from  $\{\tilde{\lambda}_k^m\}$  the later fact yields the existence of a bounded operator  $\mathcal{P}_0 : X_{\mathcal{A}} \rightarrow \mathbb{C}^n$  for which the relations (5.12), (5.13) are satisfied. Besides, this operator is unique if we require additionally:  $\mathcal{P}_0 \tilde{\varphi}_j^0 = 0$ ,  $j = 1, \dots, n$ , which is equivalent to condition (C). If we allow coincidence  $\lambda_k^m = \tilde{\lambda}_k^m$  for some indices  $\{k, m\} \in \mathbb{I}$  one needs to use continuous dependence of operators  $\mathcal{D}$  on the sequence  $\{\lambda_k^m\}$  (see Remark to Lemma 5.2). We approximate  $\{\lambda_k^m\}$  by  $\{\lambda_k'^m\}$  ( $\sum_k |\lambda_k'^m - \lambda_k^m|^2 \rightarrow 0$ ) such that  $\{\lambda_k'^m\} \neq \tilde{\lambda}_k^m$ . Since the conditions (5.12), (5.13) are satisfied for operator  $\mathcal{P}_0(\{\lambda_k'^m\})$  they are also satisfied for  $\mathcal{P}_0(\{\lambda_k^m\})$ . This completes the proof  $\square$

**Lemma 5.5.** *Let  $\{\lambda_k^m\}$  be a given sequence such that  $\sum_k |\lambda_k^m - \tilde{\lambda}_k^m|^2 < \infty$  and  $\{\hat{\alpha}_{mm_0}^{k_0}\}$  be an arbitrary sequence satisfying*

$$\sum_{k_0} |\hat{\alpha}_{mm_0}^{k_0}|^2 < \infty, \quad m \neq m_0; \quad \sum_{k_0} |\hat{\alpha}_{mm}^{k_0} - 1|^2 < \infty.$$

*Then for any  $\varepsilon > 0$  and  $m = 1, \dots, n$  there is a sequence  $\{\alpha_{mm_0}^{k_0}\}$  satisfying*

$$\sum_{k_0} |\alpha_{mm_0}^{k_0} - \hat{\alpha}_{mm_0}^{k_0}|^2 < \varepsilon$$

*and such that the operator  $D_m$  has a bounded inverse.*

*Proof.* First, for given  $\{\lambda_k^m\}$ , let us choose  $\varepsilon_0 > 0$  such that  $D_m$  will be invertible for

$$\sum_{k_0} |\alpha_{mm_0}^{k_0}|^2 < \varepsilon_0, \quad m \neq m_0, \quad \sum_{k_0} |\alpha_{mm}^{k_0} - 1|^2 < \varepsilon_0.$$

Then, one can find a great enough  $N$  such that

$$\sum_{|k_0| > N} |\hat{\alpha}_{mm_0}^{k_0}|^2 < \varepsilon_0, \quad \sum_{|k_0| > N} |\hat{\alpha}_{mm}^{k_0} - 1|^2 < \varepsilon_0.$$

Next we consider the sequences  $\{\alpha_{mm_0}^{k_0}\}$  for which

$$(5.15) \quad \alpha_{mm_0}^{k_0} = \hat{\alpha}_{mm_0}^{k_0}, \text{ as } |k_0| > N.$$

Our goal is to choose the remaining components  $\alpha_{mm_0}^{k_0}$ ,  $|k_0| \leq N$  in order to satisfy the requirement of Lemma. Denote rows of matrix  $D_m$  by  $(\ell_{k_0}^{m_0})^*$ ,  $m_0 = 1, \dots, n$ ;  $k_0 \in \mathbb{Z}$  and let  $q_{k_0}^{m_0}$  be the correspondent components of the vector  $q = D_m p$ , i.e.  $q_{k_0}^{m_0} = (\ell_{k_0}^{m_0})^* p$ ,  $p \in \ell_2^n$ ,  $m_0 = 1, \dots, n$ ;  $k_0 \in \mathbb{Z}$ . The space  $L = \underbrace{\ell_2 \times \dots \times \ell_2}_{n \text{ times}}$  may be written as  $L = L^1 \oplus L^2$ , where

$$L^1 = \{q : q_{k_0}^{m_0} = 0, \quad m_0 = 1, \dots, n, \quad |k_0| > N\},$$

and

$$L^2 = \{q : q_{k_0}^{m_0} = 0, \quad |k_0| \leq N, \quad m_0 = 1, \dots, n\}.$$

Let  $P$  be the orthogonal projector on  $L_2$ . Let us observe that the lines  $(\ell_{k_0}^{m_0})^*$  for  $|k_0| > N$  do not depend on chosen components  $\alpha_{mm_0}^{k_0}$ ,  $|k_0| \leq N$  and that if we put  $\alpha_{mm_0}^{k_0} = \delta_{mm_0}$ ,  $|k_0| \leq N$ ,  $m, m_0 = 1, \dots, n$ , then the operator  $D_m : L \rightarrow L$  has

a bounded inverse. This means that for all sequences  $\{\alpha_{mm_0}^{k_0}\}$  satisfying (5.15) we have  $PD_m L = L^2$  and the invertibility of  $D_m$  occurs if and only if

$$(5.16) \quad D_m L \supset L_1.$$

Let  $L^{1'}$  be the subspace

$$\{p \in L : (\ell_{k_0}^{m_0})^* p = 0, |k_0| > N, m_0 = 1, \dots, n\}$$

of dimension  $(2N+1)n$ . Denote by  $\ell_k^{j'}, j = 1, \dots, n; |k| \leq N$  a basis of  $L^{1'}$ . With this notations one can see that (5.16) is equivalent to the invertibility of  $(2N+1)n \times (2N+1)n$  matrix

$$M = \left\{ (\ell_{k_0}^{m_0})^* \ell_k^{j'}, |k_0| \leq N, |k| \leq N, m_0 = 1, \dots, n, j = 1, \dots, n \right\}$$

i.e.  $\det M \neq 0$ . The components of  $M$  are linear functions of chosen  $\alpha_{mm_0}^{k_0}, |k_0| \leq N$ . Therefore, its determinant is a polynomial of these coefficients. Besides,  $\det M$  is not identical zero because matrix  $D_m$  is invertible if we take  $\alpha_{mm_0}^{k_0} = \delta_{mm_0}$ , i.e.  $\det M \neq 0$ . This implies that  $M$  is invertible almost everywhere in  $\mathbb{C}^{2(N+1)n}$ . This fact completes the proof of Lemma because we can choose  $\alpha_{mm_0}^{k_0}, |k_0| \leq N$ , in such a way that

$$\sum_{|k_0| \leq N} |\alpha_{mm_0}^{k_0} - \hat{\alpha}_{mm_0}^{k_0}| < \varepsilon$$

and the operator  $D_m$  will be invertible.  $\square$

**Remark 5.6.** From the proof of Lemma 5.5 it is easy to see that actually sequence  $\alpha_{mm_0}^{k_0}$  may differ from  $\hat{\alpha}_{mm_0}^{k_0}$  only for a finite number of components  $|k_0| \leq N$ .

Due to Lemma 5.5 the formulation of Theorem 5.4 may be generalized in the following way.

**Theorem 5.7.** Let the sequence  $\{\lambda_k^m\}_{k \in \mathbb{Z}, m=1, \dots, n}$  and vectors  $z_m, m = 1, \dots, n$  be chosen according the assumptions of Theorem 5.4. Then for any sequence of vectors  $\{\hat{d}_k^m\}_{k \in \mathbb{Z}, m=1, \dots, n}$  satisfying

$$\sum_k \|\hat{d}_k^m - z_m\|^2 < \infty, m = 1, \dots, n,$$

and for any  $\varepsilon > 0$ , there is a sequence  $\{d_k^m\}_{k \in \mathbb{Z}, m=1, \dots, n}$  :

$$\sum_k \|d_k^m - \hat{d}_k^m\|^2 < \varepsilon$$

such that, for some choice of matrices  $A_{-1}, A_2(\theta), A_3(\theta)$ , satisfying  $\int_{-1}^0 A_3(\theta) d\theta = 0$ , the conditions i), ii) of Theorem 5.4 are verified. Moreover,  $\{d_k^m\}$  may be chosen in such a way, that  $d_k^m = \hat{d}_k^m$  for all  $|k| > N$  and for some  $N \in \mathbb{N}$ .

And then we obtain the following result.

**Theorem 5.8.** Let the sequences  $\{\lambda_k^m\}_{k \in \mathbb{Z}, m=1, \dots, n}$  and  $\{\hat{d}_k^m\}_{k \in \mathbb{Z}, m=1, \dots, n}$  be from Theorem 5.7. Let, in addition, the complex numbers  $\lambda_j^0, j = 1, \dots, n$  be different

from each other and different from  $\lambda_k^m$  and let  $d_j^0$ ,  $j = 1, \dots, n$  be linear independent vectors. Then, for any  $\varepsilon > 0$  there exist  $N > 0$ , a sequence  $\{d_k^m\}_{k \in \mathbb{Z}, m=1, \dots, n}$ :

$$\sum_k \|d_k^m - \widehat{d}_k^m\|^2 < \varepsilon, \quad d_k^m = \widehat{d}_k^m, \quad |k| > N, \quad m = 1, \dots, n$$

and a choice of matrices  $A_{-1}, A_2(\theta), A_3(\theta)$  such that:

- i) all the numbers  $\{\lambda_k^m\}_{k \in \mathbb{Z}, m=1, \dots, n} \cup \{\lambda_j^0\}_{j=1, \dots, n}$  are roots of the characteristic equation  $\det \Delta(\lambda) = 0$ ;
- ii)  $d_k^{m*} \Delta(\lambda_k^m) = 0$ ,  $m = 1, \dots, n$ ,  $k \in \mathbb{Z}$  and  $d_j^{0*} \Delta(\lambda_j^0) = 0$ .

*Proof.* Denote by  $C$  a  $(n \times n)$  matrix uniquely defined by the equalities:

$$d_j^{0*} C = \lambda_j^0 d_j^{0*}, \quad j = 1, \dots, n,$$

and let us put

$$\widehat{f}_k^{m*} = \widehat{d}_k^{m*} (I - \frac{1}{\lambda_k^m} C)^{-1}, \quad k \in \mathbb{Z}, \quad m = 1, \dots, n$$

if  $\lambda_k^m \neq 0$  and

$$\widehat{f}_k^{m*} = \widehat{d}_k^{m*} C^{-1}$$

for  $\lambda_k^m = 0$ .

It is easy to see that the sequences  $\{\widehat{f}_k^m\}$  are also quadratically closed to  $z_m$ :

$$\sum_k \|\widehat{f}_k^m - z_m\|^2 < \infty.$$

Therefore, due to Theorem 5.7, for any  $\varepsilon > 0$ , there exist matrices  $A_{-1}, \widehat{A}_2(\theta), \widehat{A}_3(\theta)$  ( $\int_{-1}^0 \widehat{A}_3(\theta) d\theta = 0$ ), a number  $N > 0$  and a sequence of vectors  $\{f_k^m\}_{k \in \mathbb{Z}, m=1, \dots, n}$ :

$$\sum_k \|f_k^m - \widehat{f}_k^m\|^2 < \frac{\varepsilon}{M^2}, \quad f_k^m = \widehat{f}_k^m, \quad |k| > N$$

such that

$$f_k^{m*} \widehat{\Delta}(\lambda_k^m) = 0, \quad k \in \mathbb{Z}, \quad m = 1, \dots, n;$$

where  $M = \sup\{\|I - \frac{1}{\lambda_k^m} C\|, \lambda_k^m \neq 0; \|C\|\}$  and

$$\widehat{\Delta}(\lambda) = \lambda I - \lambda e^{-\lambda} A_{-1} + \int_{-1}^0 \lambda e^{\lambda\theta} \widehat{A}_2(\theta) d\theta + \int_{-1}^0 e^{\lambda\theta} \widehat{A}_3(\theta) d\theta.$$

Now, let us put

$$\Delta(\lambda) = (I - \frac{1}{\lambda} C) \widehat{\Delta}(\lambda), \quad \lambda \neq 0.$$

One can note that  $\widehat{\Delta}(0) = 0$  and the function  $\widehat{\Delta}_1(\lambda) = \frac{1}{\lambda} \widehat{\Delta}(\lambda)$  may be extended to zero by the formula

$$\widehat{\Delta}_1(0) = I - A_{-1} + \int_{-1}^0 \widehat{A}_2(\theta) d\theta + \int_{-1}^0 \lim_{\lambda \rightarrow 0} \frac{e^{\lambda\theta} - 1}{\lambda} \widehat{A}_3(\theta) d\theta$$

Then, one can define

$$\Delta(0) = -C \widehat{\Delta}_1(0).$$

Let us observe that  $\Delta(\lambda)$  can be written as

$$\Delta(\lambda) = \lambda I - \lambda e^{-\lambda} A_{-1} + \int_{-1}^0 \lambda e^{\lambda\theta} \widehat{A}_2(\theta) d\theta + \int_{-1}^0 e^{\lambda\theta} \widehat{A}_3(\theta) d\theta - C + e^{-\lambda} C A_{-1}$$

$$\begin{aligned}
& - \int_{-1}^0 e^{\lambda\theta} C \hat{A}_2(\theta) d\theta + \int_{-1}^0 e^{\lambda\theta} \int_{-1}^{\theta} C \hat{A}_3(\tau) d\tau d\theta, \\
& = \lambda I - \lambda e^{-\lambda} A_{-1} + \int_{-1}^0 \lambda e^{\lambda\theta} A_2(\theta) d\theta + \int_{-1}^0 e^{\lambda\theta} A_3(\theta) d\theta
\end{aligned}$$

for  $A_2(\theta)$  and  $A_3(\theta)$  given by

$$\begin{aligned}
A_2(\theta) &= \hat{A}_2(\theta) - (\theta + 1)C - \theta C A_{-1}, \\
A_3(\theta) &= \hat{A}_3(\theta) + \int_{-1}^{\theta} C \hat{A}_3(\tau) d\tau - C - C A_{-1}
\end{aligned}$$

It remains to note that, with this choice of matrices, the conditions i), ii) are satisfied. Indeed:

$$\det \Delta(\lambda) = \det(I - \frac{1}{\lambda} C) \det \hat{\Delta}(\lambda),$$

so all the numbers  $\{\lambda_k^m\}$  and  $\{\lambda_j^0\}$  are roots of the characteristic equation and

$$\begin{aligned}
d_k^{m*} \Delta(\lambda_k^m) &= f_k^{m*} \hat{\Delta}(\lambda_k^m), \quad k \in \mathbb{Z}, \quad m = 1, \dots, n, \\
&= 0, \\
d_j^{0*} \Delta(\lambda_j^0) &= 0, \quad j = 1, \dots, m
\end{aligned}$$

and finally, for all  $m = 1, \dots, n$ , we have

$$\sum_k \|d_k^m - \hat{d}_k^m\|^2 \leq \sum_{k,m} M^2 \|f_k^m - \hat{f}_k^m\| < \varepsilon.$$

This completes the proof.  $\square$

## 6. CONCLUSION

We give here some conditions on sets of complex numbers  $\{\lambda\}$  and  $n$ -vectors  $\{d\}$  such that they form a spectral set for a neutral type systems. This is a first etap for solving vector moment problems using the exact controllability properties of a neutral type system related to the given moment problem.

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